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Non-commutative differential calculus and q-analysis

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Abstract. Starting from the formulation of covariant non-commutative differential calculus recently given by Wess and Zumino we construct a deformation of the Virasoro algebra, which allow us to identify the variables and differential operators on the quantum plane $\boldsymbol{R}_q^2$ to those on the classical plane $\boldsymbol{R}^2$. This correspondence indicates how non-commutative geometry can be understood in terms of q-analysis on the commutative plane. We generalize this result to the general n-dimensional case and discuss some of its consequences.

1. Introduction

In the past few years the subject of quantum groups and quantum algebras [1] has attracted the attention of mathematicians and physicists. Some disciplines in which they play an important role include conformal field theories [2], integrable systems [3], statistical models [4] and two-dimensional gravity [5], all of them having the Yang–Baxter algebra [6] as a common ground.

Another development includes the representation of quantum algebras (or q-deformed Lie algebras) in terms of q-analogues of bosonic and fermionic [7], parabose and parafermi [8] operators, and the formulation of a differential calculus on the space of a quantum group [9, 10].

In particular, in this paper our interest resides in the covariant formulation of Wess and Zumino and its relation to q-differential analysis. In section 2 we define a deformation of the Virasoro algebra in terms of one of the coordinates of the quantum plane $\boldsymbol{R}_q^2$, and based on this in section 3 we identify each first-order differential operator on the set of differentiable self-maps of the quantum plane with an infinite order differential operator on the set of smooth maps of $\boldsymbol{R}^2$. We show that these differential operators can be written as a product of a q-derivative and a scaling operator, bringing therefore the theory of q-analysis to play a corresponding role in commutative geometry. We generalize this result to the $n>2$ dimensional case and conclude with some remarks.

Let us denote by $\hat{x}_i$, $i=1,\ldots,n$, the coordinates of the real quantum n-plane, where

$$\hat{x}_i\hat{x}_j = q\hat{x}_j\hat{x}_i \qquad i<j \tag{1}$$

is invariant under $\mathrm{Gl}_q(n)$ transformations and q is a complex number. Recently, a covariant formulation has been given [10] in terms of operators acting linearly on the variables $\hat{x}_i$, and satisfying simple consistency relations with the R-matrix of $\mathrm{Gl}_q(n)$. For further reference, we recall that for $n=2$ the relations between variables and derivatives have been shown to be

$$\hat{\partial}_x\hat{\partial}_y = q^{-1}\hat{\partial}_y\hat{\partial}_x \tag{2}$$

$$\hat{\partial}_x\hat{x} = 1 + q^2\hat{x}\hat{\partial}_x + (q^2-1)\hat{y}\hat{\partial}_y \tag{3}$$

$$\hat{\partial}_x\hat{y} = q\hat{y}\hat{\partial}_x \tag{4}$$

$$\hat{\partial}_y\hat{x} = q\hat{x}\hat{\partial}_y \tag{5}$$

$$\hat{\partial}_y\hat{y} = 1 + q^2\hat{y}\hat{\partial}_y \tag{6}$$

which are easily seen to yield to commutative calculus for $q=1$.

2. Virasoro algebra on R_q^2

As is well known, the classical Virasoro algebra elements L_n have a representation in terms of differential operators given by

$$L_{-n} = y^{n+1}\partial_y \tag{7}$$

and therefore it is suggestive that a q-deformed Virasoro algebra consists of elements $\hat{L}_n$ defined as

$$\hat{L}_n = f(q)\hat{y}^{-n+1}\hat{\partial}_y \tag{8}$$

such that $f(q=1)=1$, and the choice of the variable $\hat{y}$ is obvious once one compares (6) with (3). A simple calculation with the help of relations such as

$$\hat{\partial}_y\hat{y}^{\pm n} = \hat{y}^{\pm n-1}\frac{1-q^{\pm 2n}}{1-q^2} + \hat{y}^{\pm n}q^{\pm 2n}\hat{\partial}_y \tag{9}$$

gives for $f(q)=q$ that the $\hat{L}_n$ operators satisfy the algebra

$$[\hat{L}_m, \hat{L}_n]_{q^{n-m}} \equiv q^{n-m}\hat{L}_m\hat{L}_n - q^{m-n}\hat{L}_n\hat{L}_m = [m-n]\hat{L}_{m+n} \tag{10}$$

where

$$[m-n] = \frac{q^{m-n}-q^{n-m}}{q-q^{-1}}.$$

It is clear that (10) reduces to the classical Virasoro algebra in the $q=1$ limit. In particular, the $SU(1,1)_q$ realization is given by the set of operators

$$\hat{L}_0 = q\hat{y}\hat{\partial}_y \qquad \hat{L}_1 = q\hat{\partial}_y \qquad \hat{L}_{-1} = q\hat{y}^2\hat{\partial}_y \tag{11}$$

satisfying

$$[\hat{L}_0, \hat{L}_{-1}]_{q^{-1}} = \hat{L}_{-1} \qquad [\hat{L}_1, \hat{L}_0]_{q^{-1}} = \hat{L}_1 \qquad [\hat{L}_1, \hat{L}_{-1}]_{q^{-2}} = [2]\hat{L}_0. \tag{12}$$

A realization of Witten's deformation [11] of SU(2)

$$[\hat{T}_0, \hat{T}_1]_{p^{1/2}} = \hat{T}_1 \qquad [\hat{T}_{-1}, \hat{T}_0]_{p^{1/2}} = \hat{T}_{-1} \qquad [\hat{T}_1, \hat{T}_{-1}] = \hat{T}_0 - (p^{1/2}-p^{-1/2})\hat{T}_0^2 \tag{13}$$

is written on the quantum plane as follows

$$\hat{T}_1 = \mathrm{i}p^{1/2}\hat{\partial}_y \qquad \hat{T}_0 = -p^{1/2}\hat{y}\hat{\partial}_y \qquad \hat{T}_{-1} = \mathrm{i}p^{1/2}\hat{y}^2\hat{\partial}_y \tag{14}$$

where the parameter $p=q^2$.

3. R_q^2-coordinates as mappings on R^2

In [12] it has been shown that the algebra in (10) can be realized in terms of standard differential operators as

$$\hat{L}_n = y^{-n}\frac{q^{2y\partial_y}-1}{q-q^{-1}} \tag{15}$$

and therefore comparing with (8) we see that the q-deformed Virasoro algebra yields to identify

$$q\hat{y}^{-n+1}\hat{\partial}_y \rightarrow y^{-n}\frac{q^{2y\partial_y}-1}{q-q^{-1}} \tag{16}$$

such that, by identifying the quantum plane coordinate $\hat{y}$ with the standard variable y, we obtain that the q-derivative has a representation on the commutative plane given by

$$\hat{\partial}_y \rightarrow D_q \equiv q^{-1}y^{-1}\frac{q^{2y\partial_y}-1}{q-q^{-1}} \tag{17}$$

which has the appropriate limit $\hat{\partial}_y \rightarrow \partial_y$ for $q=1$. We can also readily check that (6) is satisfied on R^2. Notice that once we identify the number operator $N_y y^n = ny^n$ with the differential operator $y\partial_y$ we see that D_q corresponds to the q-differential operator of q-analysis [13]. We can easily check that for an arbitrary function $\phi(y)$

$$D_q\phi(y) = q^{-1}y^{-1}\frac{\phi(q^2y)-\phi(y)}{q-q^{-1}} \tag{18}$$

and that to the inverse of D_q corresponds the operation

$$\begin{aligned}\int_q \mathrm{d}y\, g(y) &= -(q^2-1)y\sum_{n=0}^{\infty} q^{2n(1+y\partial_y)}g(y) + \text{constant}\\ &= -(q^2-1)y\frac{1}{1-q^{2(1+y\partial_y)}}g(y) + \text{constant}.\end{aligned} \tag{19}$$

Inspection of (2), (4) and (5) tells us that for the coordinate $\hat{x}$ we can write

$$\hat{x} \rightarrow f(x, \partial_x, \ldots, \partial_x^n, \ldots)q^{y\partial_y} \qquad \hat{\partial}_x \rightarrow g(x, \partial_x, \ldots, \partial_x^n, \ldots)q^{y\partial_y} \tag{20}$$

such that together with (3) the functions f and g are required to satisfy

$$gf - q^2fg = 1 \tag{21}$$

which is of the same functional type as (6). Therefore, we define

$$g \equiv q^{-1}x^{-1}\frac{q^{2x\partial_x}-1}{q-q^{-1}} \qquad f \equiv x. \tag{22}$$

In particular, for the q-deformation of the quantum mechanical momentum operators $\hat{p}_x$ and $\hat{p}_y$ we have the mappings

$$\hat{p}_x = -\mathrm{i}q^2\hat{\partial}_x \rightarrow -\mathrm{i}qx^{-1}\frac{q^{2x\partial_x}-1}{q-q^{-1}}q^{y\partial_y} \qquad \hat{p}_y = -\mathrm{i}q\hat{\partial}_y \rightarrow -\mathrm{i}y^{-1}\frac{q^{2y\partial_y}-1}{q-q^{-1}} \tag{23}$$

and, as we can see, if we want to describe a local theory defined on $\boldsymbol{R}_q^2$ through its correspondence in the classical plane we are forced to deal with a non-local theory on $\boldsymbol{R}^2$.

A representation of the well known q-deformation of the Lie algebra of SU(2)

$$[J_3, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = [2J_3] \tag{24}$$

can be given in terms of the operators $D_q^{(i)} \equiv q^{-1} x_i^{-1}(q^{2x_i\partial_i} - 1)/(q - q^{-1})$ by defining

$$\begin{aligned} &J_+ = qq^{-N_2/2} x_2 D_q^{(1)} q^{-N_1/2} \qquad J_- = qq^{-N_1/2} x_1 D_q^{(2)} q^{-N_2/2} \\ &J_3 = \tfrac{1}{2}(N_2 - N_1). \end{aligned} \tag{25}$$

The case $n = 3$ can be similarly work out. From [10] we learned that for $n = 3$ we have that

$$\hat{\partial}_x \hat{x} = 1 + q^2 \hat{x} \hat{\partial}_x + (q^2 - 1)\hat{y}\hat{\partial}_y + (q^2 - 1)\hat{z}\hat{\partial}_z \tag{26}$$

$$\hat{\partial}_y \hat{y} = 1 + q^2 \hat{y} \hat{\partial}_y + (q^2 - 1)\hat{z}\hat{\partial}_z \tag{27}$$

$$\hat{\partial}_z \hat{z} = 1 + q^2 \hat{z} \hat{\partial}_z \tag{28}$$

and following the previous procedure we find that the $\hat{x}$, $\hat{y}$ and $\hat{z}$ coordinates correspond to the following self-maps of $\boldsymbol{R}^3$

$$\begin{aligned} &\hat{z} \to z \qquad \hat{\partial}_z \to q^{-1} z^{-1} \frac{q^{2z\partial_z} - 1}{q - q^{-1}} \\ &\hat{y} \to y q^{z\partial_z} \qquad \hat{\partial}_y \to q^{-1} y^{-1} q^{z\partial_z} \frac{q^{2y\partial_y} - 1}{q - q^{-1}} \\ &\hat{x} \to x q^{y\partial_y} q^{z\partial_z} \qquad \hat{\partial}_x \to q^{-1} x^{-1} q^{y\partial_y} q^{z\partial_z} \frac{q^{2x\partial_x} - 1}{q - q^{-1}} \end{aligned} \tag{29}$$

from where we can easily generalize to the case of arbitrary n. We should also remark that these differential mappings are a representation of the φ and $\varphi^\dagger$ operators discussed in the last reference in [7]. Basically, one identifies†: $\hat{x}_i \to \varphi_i^\dagger$ and $\hat{\partial}_i \to \varphi_i$, $i = 1, \ldots n$.

Then, the transformations obtained here give us the relation between differential operators on $\boldsymbol{R}_q^n$ and those on the classical n-plane, and could be useful to describe the dynamics of an n-dimensional system on $\boldsymbol{R}_q^n$ in terms of commutative geometry. In particular, the identification in (29) indicates that the action of differential operators $\hat{\partial}_i$ on functions in $\boldsymbol{R}_q^n$ corresponds to the action of the differential operator of q-analysis times scaling operators on functions in $\boldsymbol{R}^n$. Therefore, functions of non-commutative variables can be understood in terms of the theory of q-hypergeometric functions and q-series. One important implication of this correspondence is that several aspects of q-analysis, which were well studied by mathematicians during the first half of this century [14], give a new angle to approach non-commutative geometry.

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